Homework 4 Solutions

1. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be a set of vectors in $\mathbb{R}^n$.

   (a) Recall that $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly independent if and only if there is a pivot in each column of the matrix $\begin{bmatrix}\mathbf{v}_1 & \cdots & \mathbf{v}_p\end{bmatrix}$. Explain why this is true.

   By definition, $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly independent if $x_1\mathbf{v}_1 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$ only when $x_1 = x_2 = \cdots = x_n = 0$. Since $x_1\mathbf{v}_1 + \cdots + x_p\mathbf{v}_p = A\mathbf{x}$, where $A = \begin{bmatrix}\mathbf{v}_1 & \cdots & \mathbf{v}_p\end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, this is equivalent to saying that $A\mathbf{x} = \mathbf{0}$ only has the zero solution. This occurs exactly when there are no free variables; in other words, when A has a pivot in every column.

   (b) Recall that $\mathbf{b} \in \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ if and only if $A\mathbf{x} = \mathbf{b}$ has a solution, where $A = \begin{bmatrix}\mathbf{v}_1 & \cdots & \mathbf{v}_p\end{bmatrix}$. Explain why this is true.

   By definition, $\mathbf{b} \in \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ if $\mathbf{b}$ is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_p$; that is, $\mathbf{b} = x_1\mathbf{v}_1 + \cdots + x_p\mathbf{v}_p$ for some scalars $x_1, \ldots, x_p \in \mathbb{R}$. Since $x_1\mathbf{v}_1 + \cdots + x_p\mathbf{v}_p = A\mathbf{x}$, where $A = \begin{bmatrix}\mathbf{v}_1 & \cdots & \mathbf{v}_p\end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, this is equivalent to saying that $A\mathbf{x} = \mathbf{b}$ has a solution.

2. Let $\mathbf{v}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix}1\\1\\0\end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix}1\\1\\1\end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix}3\\4\\5\end{bmatrix}$.

   (a) Show that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$ by showing that the set is linearly independent and spans $\mathbb{R}^3$.

   Since the matrix $\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3\end{bmatrix} = \begin{bmatrix}1 & 1 & 1\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix}$ has a pivot in each column, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

   To show that the set spans $\mathbb{R}^3$, we need to show that every $\mathbf{b} \in \mathbb{R}^3$ is in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. This is equivalent to showing that the system $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b}$, where $A = \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3\end{bmatrix}$. Since A has a pivot in every row, as seen above, $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b}$. Hence the set spans $\mathbb{R}^3$.

(b) Explain why $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly dependent.

Since the matrix $\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4\end{bmatrix}$ has four columns, but only 3 rows, it cannot have a pivot in every column. Thus $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly dependent.

(c) Express $\mathbf{v}_3$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_4$ by solving a related system.

We must solve $\begin{bmatrix}1\\1\\1\end{bmatrix} = x_1\begin{bmatrix}1\\0\\0\end{bmatrix} + x_2\begin{bmatrix}1\\1\\0\end{bmatrix} + x_3\begin{bmatrix}3\\4\\5\end{bmatrix}$,
or equivalently, $\begin{bmatrix}1 & 1 & 3\\0 & 1 & 4\\0 & 0 & 5\end{bmatrix}\mathbf{x} = \begin{bmatrix}1\\1\\1\end{bmatrix}$. We can do this by reducing the augmented matrix to RREF or by backward substitution. We'll use backward substitution. Rewriting the system gives:

$$\begin{aligned} x_1 + x_2 + 3x_3 &= 1\\ x_2 + 4x_3 &= 1\\ 5x_3 &= 1 \end{aligned}$$

Performing backward substitution gives the solution $x_1 = \frac{1}{5}, x_2 = \frac{1}{5}, x_3 = \frac{1}{5}$. We thus have:

$$\mathbf{v}_3 = \frac{1}{5}\mathbf{v}_1 + \frac{1}{5}\mathbf{v}_2 + \frac{1}{5}\mathbf{v}_4.$$

3. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^3$ such that $\mathbf{v}_2 = 2\mathbf{v}_1 + \mathbf{v}_3$. Show that $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_3\}$ by doing the following:

(a) Show that if $\mathbf{v} \in \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, then $\mathbf{v} \in \text{Span}\{\mathbf{v}_1, \mathbf{v}_3\}$

Let $\mathbf{v} \in \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. then $\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2$. Since $\mathbf{v}_2 = 2\mathbf{v}_1 + \mathbf{v}_3$, we have that $\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = x_1\mathbf{v}_1 + x_2(2\mathbf{v}_1 + \mathbf{v}_3) = (x_1 + 2x_2)\mathbf{v}_1 + x_2\mathbf{v}_3 \in \text{Span}\{\mathbf{v}_1, \mathbf{v}_3\}$.

(b) Show that if $\mathbf{v} \in \text{Span}\{\mathbf{v}_1, \mathbf{v}_3\}$, then $\mathbf{v} \in \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$

Let $\mathbf{v} \in \text{Span}\{\mathbf{v}_1, \mathbf{v}_3\}$. then $\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_3$. Since $\mathbf{v}_2 = 2\mathbf{v}_1 + \mathbf{v}_3$, we have that $\mathbf{v}_3 = -2\mathbf{v}_1 + \mathbf{v}_2$; hence $\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_3 = x_1\mathbf{v}_1 + x_2(-2\mathbf{v}_1 + \mathbf{v}_2) = (x_1 - 2x_2)\mathbf{v}_1 + x_2\mathbf{v}_2 \in \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

(c) Deduce that $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_3\}$.

We have shown that every vector in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ belongs to $\text{Span}\{\mathbf{v}_1, \mathbf{v}_3\}$ and every vector in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_3\}$ belongs to $\text{Span}\{\mathbf{v}_1, \mathbf{v}_1\}$. That is, these sets contain the same exact vectors. Thus $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_3\}$.

4. Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \\ 0 & 1 & 4 \\ -2 & 1 & 3 \end{bmatrix}$. Determine conditions on the entries of $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ that ensures $\mathbf{b} \in C(A)$.

$\mathbf{b} \in C(A)$ if and only if $A\mathbf{x} = \mathbf{b}$ has a solution. Hence we must reduce the augmented matrix $\begin{bmatrix} A & | & \mathbf{b} \end{bmatrix}$ and decide for which values of b_1, b_2, b_3, and b_4, there is a pivot in the final column.

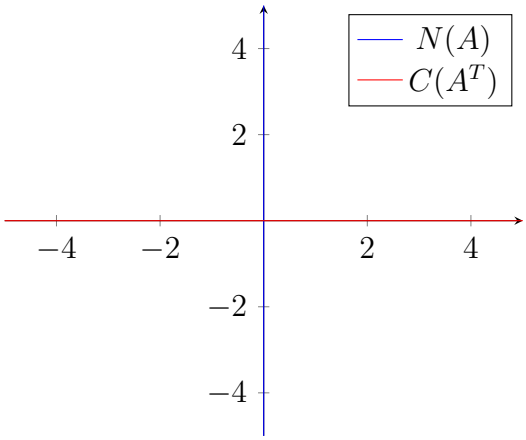
$$\begin{bmatrix} A & | & \mathbf{b} \end{bmatrix} = \left[\begin{array}{ccc|c} 1 & -1 & 2 & b_1 \\ 0 & 1 & 3 & b_2 \\ 0 & 1 & 4 & b_3 \\ -2 & 1 & 3 & b_4 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & b_1 + b_2 + 5b_3 \\ 0 & 1 & 0 & 4b_2 - 3b_3 \\ 0 & 0 & 1 & -b_2 + b_3 \\ 0 & 0 & 0 & 2b_1 + 11b_2 - 10b_3 + b_4 \end{array}\right]$$

There is a pivot in the final column if and only if $2b_1 + 11b_2 - 10b_3 + b_4 \neq 0$. Hence $\mathbf{b} \in C(A)$ if and only if $2b_1 + 11b_2 - 10b_3 + b_4 = 0$.

5. Let $A = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$.

(a) Determine $N(A)$ and $C(A^T)$ as spans and draw them in the same plane.

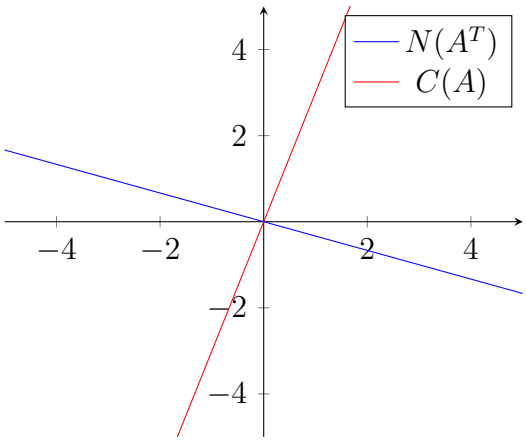
The RREF of A is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Hence x_1 is the pivot variable and x_2 is the free variable. The general solution to $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Hence $N(A) = \text{Span}\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\}$. The row space is $C(A^T) = \text{Span}\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix}\} = \text{Span}\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\}$. These subspaces are drawn below.



(b) Detemrine $C(A)$ and $N(A^T)$ as spans and draw them in the same plane.

The column space is $C(A) = \text{Span}\{\begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}\} = \text{Span}\{\begin{bmatrix}1\\3\end{bmatrix}\}$.

To determine the left nullspace, we consider the RREF of A^T, which is simply $A^T = \begin{bmatrix}1 & 3\\0 & 0\end{bmatrix}$. Rewriting the system in terms of the free variable x_2 gives $x_1 = -3x_2$. Hence the general solution to $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \begin{bmatrix}-3x_2\\x_2\end{bmatrix} = x_2\begin{bmatrix}-3\\1\end{bmatrix}$. Thus $N(A^T) = \text{Span}\{\begin{bmatrix}-3\\1\end{bmatrix}\}$. These subspaces are drawn below.



6. $A = \begin{bmatrix}1 & 2 & 3 & 4 & 5\\6 & 7 & 8 & 9 & 10\\10 & 9 & 8 & 7 & 6\\5 & 4 & 3 & 2 & 1\end{bmatrix}$

(a) Write down the RREF of A and determine rank$A$.

The RREF is $\begin{bmatrix} 1 & 0 & -1 & -2 & -3 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.
Since there are two pivot columns, the rank of A is 2.

(b) Determine the dimensions of the four fundamental subspaces (without finding a basis for each).

Since $\dim C(A) = \text{rank}A$, we have that $\dim C(A) = 2$. Since $\text{rank}A = \text{rank}(A^T)$, we have that $\dim C(A^T) = 2$. Since A has five columns, by the rank-nullity theorem, $\dim N(A) = 5 - 2 = 3$. Since A has four rows, by the rank-nullity theorem, $\dim N(A^T) = 4 - 2 = 2$.

(c) Find bases for the column space, null space, and row space of A.

The pivot columns of A form a basis for $C(A)$; hence a basis is:

$$\left\{ \begin{bmatrix} 1 \\ 6 \\ 10 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ 9 \\ 4 \end{bmatrix} \right\}.$$

The pivot rows of the RREF form a basis for $C(A^T)$; hence a basis is:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ -2 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right\}.$$

To find a basis for $N(A)$, we must solve $A\mathbf{x} = \mathbf{0}$. Rewriting the system corresponding to the RREF matrix in terms of the free variables x_3, x_4, x_5 gives

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_3 + x_4 + x_5 \\ -2x_3 - 3x_4 - 4x_5 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Hence a basis is

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

7. In class, we mentioned that if A is a matrix and R is the RREF, then the row space of A is the same as the row space of R. It is not true, however, that the column space of A is the same as the column space of R. Let's show this with an example.

Let $A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$.

(a) Find the RREF R of A.

$$R = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

(b) Write the free column of R as a linear combination of the pivot columns of R. What is the dimension of $C(R)$?

$$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

(c) Write the free column of A as a linear combination of the pivot columns of A. What is the dimension of $C(A)$?

We simply need to use the same weights as in part (a):

$$\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}.$$

(d) Show that $C(A) \neq C(R)$.

$C(A)$ is spanned by the pivot columns of A and $C(R)$ is spanned by the pivot columns of R. That is,

$$C(A) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \right\} \qquad C(R) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Since every vector in $C(R)$ is a linear combination of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, every vector in $C(R)$ must have third entry equal to zero. Hence $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \notin C(R)$. However,

$\begin{bmatrix}1\\0\\1\end{bmatrix} \in C(A)$. Thus $C(A) \neq C(R)$.

8. Let V and W be subspaces of $\mathbb{R}^n$. Suppose V has basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ and W has basis $\{\mathbf{w}_1, \ldots, \mathbf{w}_q\}$.

   (a) Show that if $\{\mathbf{v}_1, \ldots, \mathbf{v}_p, \mathbf{w}_1, \ldots, \mathbf{w}_q\}$ is linearly dependent, then $V \cap W \neq \{\mathbf{0}\}$ (Hint: recall that a set is linearly dependent if and only if one of the vectors is a linear combination of the rest. Assume that $\mathbf{w}_q$ is a linear combination of the rest and construct a nonzero vector in $V \cap W$).

   Since $\{\mathbf{v}_1, \ldots, \mathbf{v}_p, \mathbf{w}_1, \ldots, \mathbf{w}_q\}$ is linearly dependent, one of the vectors is a linear combination of the rest. Assume that $\mathbf{w}_q$ is a linear combination of the others. That is:

   $$\mathbf{w}_q = x_1\mathbf{v}_1 + \cdots + x_p\mathbf{v}_p + y_1\mathbf{w}_1 + \cdots + y_{q-1}\mathbf{w}_{q-1}$$

   Moving the vectors in W to the left, we have

   $$\mathbf{w}_q - y_1\mathbf{w}_1 - \cdots - y_{q-1}\mathbf{w}_{q-1} = x_1\mathbf{v}_1 + \cdots + x_p\mathbf{v}_p \in V$$

   Hence the vector $\mathbf{w}_q - y_1\mathbf{w}_1 - \cdots - y_{q-1}\mathbf{w}_{q-1}$ is in both W (since it is linear combination of the basis vectors of W) and in V (since it is also a linear combination of the basis vectors of V). Thus it is in $V \cap W$ and so $V \cap W \neq \{0\}$.

   (b) Show that if $\dim V + \dim W > n$, then $V \cap W \neq \{\mathbf{0}\}$.

   Since $\dim V = p$ and $\dim W = q$, we have that $p+q > n$. Thus the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p, \mathbf{w}_1, \ldots, \mathbf{w}_q\}$ is linearly dependent since the matrix $\begin{bmatrix}\mathbf{v}_1 & \ldots & \mathbf{v}_p & \mathbf{w}_1 & \ldots & \mathbf{w}_q\end{bmatrix}$ has more columns than rows and as a result cannot have a pivot in each column. Now by part (a), $V \cap W \neq \{\mathbf{0}\}$.